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An Approach to the Optimization and Approximation of Solutions of any Operator Inclusion

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Abstract. In this paper we shall present the optimization problem in a set of solutions of operator inclusion with the maximal monotone operator. Then we treat optimization problem by Galerkin method and we prove convergence of optimal values of approximated optimization problems to the one for the original problem. Finally, we apply the results to give a simple example.

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1. Introduction

The operator inclusions have a wide spectrum of applications in various branches of both pure and applied sciences, particularly in physics, economics and engineering. The theory of operator inclusions has been developed over the last years in intimate connection with physical applications in elasticity and modern plasticity theory, hydrodynamics etc. Problems connected with inclusions or with inclusions and optimization were considered by many authors, among others by N.U. Ahmed and X. Xiang [1], V. Barbu [2], K. Dimling [3], S. Hu and N. Papageorgiou [5], V.I. Ivanenko and V.S. Melnik [7], M. Kisielewicz [8] and M.A. Noor [11]. In [6] the authors employ standard Galerkin method to obtain a sequence of approximating multivalued systems and prove the existence of periodic solutions. In the work [9] the authors consider the well-posedness for family of variational inequalities and for an optimization problem with constraints defined by variational inequalities having unique solution. In [10] the authors develop the methods for the case of alone variational problems with multivalued monotone and coercive operators.

In our paper we consider the optimisation problem in the set of solutions of some inclusion with a maximal monotone operator. We apply the standard Galerkin technique (see [6] and [12]) to obtain the sequence of variational inequalities.

Next we consider the optimization of this problem in the finite-dimensional spaces. To prove the convergence of the sequence of approximated solutions we use the regularization method.

The problem considered in this paper is the generalisation of the problem for a system governed by single-valued operator from our work [4].

Let Y be a real, reflexive Banach space with the norm $\|\cdot\|$. By Y^* we denote its dual with the duality relation $\langle\cdot,\cdot\rangle$ between Y^* and Y.

Let us consider the mapping $A : Y \to 2^{Y^*}$ and the functional $J : Y \to \mathbb{R}$. We shall consider the following nonlinear inclusion

$$f \in Ay \tag{1.1}$$

for a given $f \in Y^*$.

DEFINITION 1.1. The operator A is coercive with respect to the given fixed $f \in Y^*$ if there exists an $r > 0$ such that

$$\langle y^* - f, y\rangle > 0 \quad \forall (y, y^*) \in G(A) \quad \text{with } \|y\| > r$$

where $G(A)$ is the graph of the operator A.

THEOREM 1.1. *If the operator A is maximal monotone and coercive with respect to the given fixed $f \in Y^*$ then the set Y_{ad} of solutions of the inclusion (1.1) is non-empty, convex and closed in Y, (see [12]).*

Notice that the inclusion (1.1) is equivalent to a variational inequality (see [12]):

$$\langle f - z^*, y - z\rangle \geqslant 0 \quad \forall (z, z^*) \in G(A). \tag{1.2}$$

PROBLEM P: We shall consider the following optimization problem: find $y^0 \in Y_{ad}$ such that

$$J(y^0) = \inf_{y \in Y_{ad}} J(y).$$

THEOREM 1.2. *If the functional J is continuous, strictly convex, coercive (i.e. $\lim_{\|y\|\to\infty} J(y) = \infty$) and the operator A is maximal monotone, coercive with respect to the given fixed $f \in Y^*$ then the optimization problem P has a unique solution $y^0 \in Y_{ad}$.*

The proof is immediate because it is known that any continuous strictly convex functional is weakly lower semicontinuous and the closed, convex set Y_{ad} is weakly closed in Y.

2. Regularizing and approximating optimization problem

Consider a family of regularized inclusions

$$f \in Ay_n + \varepsilon_n By_n \quad \text{for } n = 1, 2, \dots \tag{2.1}$$

$y_n \in Y$, $\varepsilon_n > 0 \quad \forall n \in \mathbb{N}$ and $\varepsilon_n \to 0$. Moreover single-valued operator $B : Y \to Y^*$.

DEFINITION 2.1. The operator $B : Y \to Y^*$ has the S-property if for every sequence $(v_n)_{n\in\mathbb{N}} \subset Y$ with the properties:

$$v_n \underset{n\to\infty}{\rightharpoonup} v(\text{weakly in } Y) \text{ and } \langle Bv_n - Bv, v_n - v\rangle \underset{n\to\infty}{\longrightarrow} 0$$

we have $v_n \underset{n\to\infty}{\longrightarrow} v$ (strongly in Y).

LEMMA 2.1. *If the operator A is maximal monotone, coercive with respect to the given fixed $f \in Y^*$ and the operator B is strictly monotone, hemicontinuous, bounded and satisfies S-property then for each $n \in \mathbb{N}$ the equation (2.1) has a unique solution y_n and $y_n \to y$ (strongly in Y) where y solves the original inclusion (1.1) and, at the same time, y is the unique solution of the variational inequality*

$$\langle By, z - y\rangle \geqslant 0 \quad \forall z \in Y_{ad} \tag{2.2}$$

(see [12]).

Let $\{V\}$ denote the family of all finite-dimensional subspaces of the original space Y i.e. $V \in \{V\}$ implies $V \subset Y$, dim $V < \infty$ and $\overline{\bigcup_{V\in\{V\}} V} = Y$.

We fix $V \in \{V\}$. In the place of the variational inequality (1.2) we consider the approximate inequality

$$\langle f - z^*, y_V - z\rangle \geqslant 0 \quad \text{for all } (z, z^*) \in G(A) \text{ and } y_V, z \in V. \tag{2.3}$$

THEOREM 2.1. *If the operator A is maximal monotone and coercive with respect to the given $f \in Y^*$ then the set V_{ad} of solutions of the inequality (2.3) is non-empty convex and closed.*

Proof. Let us denote by I_V the embedding operator from V to Y and by I_V^* the operator adjoint to I_V from Y^* to V^*. The inequality (2.3) can be presented by the operator form

$$f_V \in A_V y_V \tag{2.4}$$

where the operator $A_V = I_V^* A I_V$ and $f_V = I_V^* f$.

The operator A_V is maximal monotone and coercive with respect to f_V. Indeed, the operator A is monotone that the following inclusion is true

$$\langle y_V^* - z_V^*, I_V y_V - I_V z_V\rangle_{Y^*\times Y} \geqslant 0 \quad \forall y_V, z_V \in V \tag{2.5}$$

where $y_V^* \in A(I_V y_V)$ and $z_V^* \in A(I_V z_V)$. From the definition of the operator adjoint and (2.5) it follows that

$$\begin{aligned}&\langle y_V^* - z_V^*, I_V y_V - I_V z_V\rangle_{Y^*\times Y}\\ &\quad = \langle I_V^* y_V^* - I_V^* z_V^*, y_V - z_V\rangle_{V^*\times V} \geqslant 0 \quad \forall y_V, z_V \in V.\end{aligned}$$

Therefore the operator A_V is monotone.

From the maximal monotonicity of the operator A we have

$$\langle y^* - z^*, I_V y_V - I_V z_V\rangle_{Y^*\times Y} \geqslant 0 \quad \forall (I_V z_V, z^*) \in G(A)$$

it follows that $(I_V y_V, y^*) \in G(A)$.

It is obvious that

$$\begin{aligned}\langle I_V^* y^* - I_V^* z^*, y_V - z_V\rangle_{V^*\times V} &= \langle y_V^* - z_V^*, y_V - z_V\rangle_{V^*\times V}\\ &\geqslant 0 \quad \forall (z_V, z_V^*) \in G(A_V).\end{aligned}$$

Therefore $(y_V, y_V^*) \in G(A_V)$ where $y_V^* = I_V^* y^* \in A_V y_V$ and $z_V^* = I_V^* z^* \in A_V z_V$.

By the definition of coercivity with respect to f of the operator A we have $\langle y^* - f, I_V y_V\rangle_{Y^*\times Y} \geqslant 0 \quad \forall (I_V y_V, y^*) \in G(A)$ with $\|I_V y_V\|_Y > r$. So, we have $\langle y_V^* - f_V, y_V\rangle_{V^*\times V} > 0 \quad \forall (y_V, y_V^*) \in G(A_V)$ and $\|y_V\|_V > r$ where $y_V^* = I_V^* y^* \in A_V y_V$ and $f_V = I_V^* f$.

We conclude from this that the operator A_V is maximal monotone and coercive with respect to f_V. Therefore the set V_{ad} of the solutions of inequality (2.3) is non-empty, convex and closed. □

We shall consider the following approximate Problem P_V in the subspace V.

PROBLEM P_V: Find $y_V^0 \in V$ such that $J(y_V^0) = \inf_{y_V \in V_{ad}} J(y_V)$. Similarly as for the Problem P we can state that the Problem P_V has a unique solution $y_V^0 \in V_{ad}$.

Consider a family of regularized inclusions for the appropriate inclusion (2.4) in the form

$$f_V \in A_V y_{nV} + \varepsilon_n B_V y_{nV} \quad \text{for } n = 1, 2, \ldots \tag{2.6}$$

where $\varepsilon_n > 0 \forall n \in \mathbb{N}$ and $\varepsilon_n \to 0$ and $n \to \infty$ and $B_V = I_V^* B I_V$.

Let us now consider the problem of convergence of the approximation.

LEMMA 2.2. *If the operator A is maximal monotone, coercive with respect to the given fixed $f \in Y^*$ then for all $y \in Y_{ad}$ there exists a sequence (y_V) such that $y_V \in V_{ad}$ and $y_V \to y$ (strongly in Y) as* $\dim V \to \infty$.

Proof. Because the operator B is strictly monotone hemicontinuous bounded and satisfies S-property then the regularized inclusion (2.6) has the unique solution $\in V$ for $n = 1, 2, \ldots$ The sequence (y_{nV}) is strongly convergent to any $\bar{y}_V$ as $\varepsilon_n \to 0$. For all fixed $n \in \mathbb{N}$ from Galerkin approximation a sequence (y_{nV}) of the solutions of (2.6) is strongly convergent to any $\bar{y}_n \in Y$ as $\dim V \to \infty$ and $\bar{y}_n$ satisfies of (2.1).

From Lemma 2.1 we have that $\bar{y}_n \to \bar{y}$ (strongly in Y) and $\bar{y}$ is unique solution of (2.2). From this and the inequality

$$\|\bar{y}_V - \bar{y}\| \leqslant \|\bar{y}_V - y_{nV}\| + \|y_{nV} - \bar{y}_n\| + \|\bar{y}_n - \bar{y}\|$$

it follows that $\bar{y}_V \to \bar{y}$ (strongly in Y).

The operator B can be selected arbitrarily, so we can deduce that the sequence $(\bar{y}_V)$ exists for every $\bar{y} \in Y_{ad}$. □

Now we shall prove that the sequence (y_V^0) of solutions of problem P_V is convergent to the solution of the original problem.

THEOREM 2.2. *Let the operator A be maximal monotone, coercive with respect to the given fixed $f \in Y^*$. Let the functional J be continuous, strictly convex, coercive then a sequence (y_V^0) of solutions of the problems P_V is weakly convergent in Y to the solution y^0 of the problem P.*

Proof. Since J is continuous and coercive in reflexive Banach space then there exists a positive constant $M < \infty$ such that $\|y_V^0\| \leqslant M \quad \forall V \in \{V\}$. It follows that there exists a subsequence which will be also denoted (y_V^0) weakly convergent to $\bar{y}^0$ in Y as $\dim V \to \infty$. From Lemma 2.2 exists a sequence y_V such that $y_V \in V_{ad}$ and $y_V \to y^0$ (strongly in Y) as $\dim V \to \infty$. Because J is weakly lower semi-continuous we obtain

$$J(\bar{y}^0) \leqslant \liminf J(y_V^0) \leqslant \liminf J(y_V) = J(y^0)$$

as $\dim V \to \infty$. From the definition of y^0 it follows that $\bar{y}^0 = y^0$ and from the uniqueness of the solution of the optimization problem P not only a subsequence but the whole sequence (y_V^0) is weakly convergent to y^0 in Y as $\dim V \to \infty$. □

3. An example

In this subsection we use the results from the previous sections to present here one selected result.

A typical functional appearing in optimization problems is the quadratic functional:

$$J(y) = \|E(y - y_d)\|_H^2$$

where $E \in L(Y, H)$, $Y = H^1(\Omega)$, $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a C^2-boundary Γ (see [12]), y_d is given element of $H^1(\Omega)$.

Let E be embedding operator from Y into H. The cost functional is equivalent to:

$$J(y) = \int_{\Omega} (y(x) - y_d(x))^2 \, dx. \tag{3.1}$$

Given $f \in L^2(\Omega)$ consider the following nonlinear Neumann problem

$$\begin{cases} Fy(x) + \beta(y(x)) \ni f(x) \text{ a.e. on } \Omega \\ \dfrac{\partial y}{\partial \eta} = 0 \text{ on } \Gamma \end{cases} \tag{3.2}$$

where $F : Y \to Y^*$ is defined as

$$Fy(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial y(x)}{\partial x_j}\right) + a_0(x)y$$

$a_0, a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, 2, \ldots, n$.

We assume that:

(i) $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{n} \xi_i^2 \quad \forall \xi_i, \xi_j \in \mathbb{R}$ for certain $\alpha \geqslant 0$ and $a_0(x) \geqslant 0$.

(ii) $\beta : Y \to 2^{Y^*}$ is a coercive maximal monotone map.

(iii) $\frac{\partial y}{\partial \eta}$ is a derivative of y in the direction of the exterior normal to Γ.

From the assumptions (i), (ii) we conclude that the operator $F + \beta$ is maximal monotone on the

$$D(F+\beta) = \left\{ y \in H^1(\Omega);\, \left.\frac{\partial y}{\partial \eta}\right|_{\Gamma} = 0 \right\}$$

and $F + \beta$ is coercive with respect to the given fixed $f \in L^2(\Omega)$.

Therefore the set Y_{ad} of solutions of the differential inclusion (3.2) is non-empty, convex and closed in $L^2(\Omega)$.

The functional (3.1) is continuous, strictly convex, coercive. Therefore, from Theorem 1.2 the optimization problem for the functional (3.1) and for the differential inclusion (3.2) has a unique solution $y^0 \in Y_{ad}$. After approximation we transform our problem into a problem of mathematical programming.

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